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SOME PROPERTIES OF CLOSURE-SEQUENTIAL APPROACH SPACES

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Abstract. In this paper we introduce the category of closure-sequential approach spaces, **CSEQ**. And study some properties of closure-sequential approach space.

1. Introduction

In ([9]), we introduced the measure of countable compactness in the category **AP** of approach spaces and contractions and investigated some invariance properties of measure of countable compactness with regard to image and product. In approach theory, countability will still play a role as far as topological spaces are concerned because topological spaces are nicely embedded as a simultaneously concretely reflective and coreflective subconstruct of **AP**.

In $|\mathbf{TOP}|$, $(X, \Im)$ is called *closure-sequential* if every *sequential neighborhood* V, of a point x, is a neighbourhood of that point (V is a *sequential neighbourhood* of x if whenever $\{x_n\}_{n\in\mathbb{N}} \to x$, $\{x_n\}_{n\in\mathbb{N}} \in V$ eventually).
Clearly each first-countable space, and hence each metric space and each discrete space, is closure-sequential and they fall into the class of topological spaces which can be given completely by the knowledge of their convergent sequences.

In the paper we will define the category of closure-sequential approach spaces and will prove the category of closure-sequential approach spaces is embedded as a concretely coreflective subconstruct of **AP**. And we will show that for a closure-sequential approach space, the measures of sequential compactness and countable compactness coincide.

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2. Preliminaries

We shall use the following symbols :

$$\mathbb{R}_+ := [0,\infty[,\ \ \mathbb{R}_+^* :=]0,\infty[,\ \ \overline{\mathbb{R}_+} := [0,\infty].$$

$\mathbb{F}(X)$ will stand for the set of all filters on X, and $\mathbb{U}(X)$ will stand for the set of all ultrafilters on X. If $\mathcal{F}$ is a given filter on X, then we will denote by $\mathbb{F}(\mathcal{F})$ the collection of all filters on X which are finer than $\mathcal{F}$, and by $\mathbb{U}(\mathcal{F})$ the collection of all ultrafilters on X which are finer than $\mathcal{F}$.

We recall that, in an approach space X, the adherence operator is defined as

$$\alpha\mathcal{F} \doteqdot \sup_{F\in\mathcal{F}} \delta(x,F), \qquad \forall x\in X,\ \ \forall\mathcal{F}\in\mathbb{F}(X)$$

where $\delta : X\times 2^X \to \overline{\mathbb{R}_+}$ is the distance on X determining the approach structure.

Finally, if $|\mathbf{SET}|$ is the class of all sets and $X\in|\mathbf{SET}|$, we shall denote the set of all finite (resp. countable) subsets of X by $2^{(X)}$ (resp. $2^{((X))}$).

We recall also that a filter $\mathcal{F}$ on X is called *countable* if it has a filter base with a countable number of elements.

By $\mathbb{F}c(X)$(resp. $\mathbb{F}_e(X)$) we denote the countable (resp. elementary) filters on X.

For an approach space the measure of countable compactness is defined as

$$\mu_{cc}(X) := \sup_{\psi\in\Psi}\ \inf_{K\in 2^{(\Gamma_\psi)}}\ \sup_{z\in X}\ \inf_{k\in K} \psi(k)(z)$$

where $\Psi = \{\psi : \Gamma_\psi\subset\mathbb{N}\to\cup_{x\in X}\mathcal{A}(x) \mid \forall x,\ \exists n\in\Gamma_\psi : \psi(n)\in\mathcal{A}(x)\}$.

For an approach space the measure of sequential compactness is defined as

$$\mu_{sc}(X) = \sup_{(x_n)_{n\in\mathbb{N}}\in r(X)}\ \inf_{k\uparrow:\mathbb{N}\to\mathbb{N}}\ \inf_{x\in X} \lambda < x_{k(n)} > (x).$$

If X is an approach space then the measure of Lindelöf of X is defined as

$$L(X) = \sup_{\phi\in\prod_{z\in X}\Phi(z)}\ \inf_{Y\in 2^{((X))}}\ \sup_{z\in X}\ \inf_{y\in Y} \phi(y)(z)$$

and for the $p\mathbf{MET}^\infty$ space (X,d) we have

$$L(X) = \inf_{Y\in 2^{((X))}}\ \sup_{z\in X}\ \inf_{y\in Y} d(y,z)$$

and, equivalently [2],

$$L(X) = \sup_{\mathcal{F}\in\mathbb{F}_w(X)}\ \inf_{x\in X} \alpha\mathcal{F}(x) = \sup_{\mathcal{F}\in\mathbb{F}_w(X)}\ \inf_{x\in X}\ \sup_{F\in\mathcal{F}}\ \inf_{y\in F} d(x,y)$$

where $\mathbb{F}_w$ stands for the set of filters with the countable intersection property.

In [9], for an extended pseudometric approach space X, we have,

$$\mu_{sc}(X) = \mu_{cc}(X) = L(X) = \mu_c(X).$$

3. Closure-Sequential Spaces

Now, in the frame of **AP** we begin with some definitions.

DEFINITION 3.1. An approach space $(X, (\mathcal{A}(x))_{x\in X})$ shall be called *closure-sequential* if for each $\mu \in [0,\infty]^X$ and each $x \in X$,

$$\mu \in \mathcal{A}(x) \iff \{x_n\}_{n\in\mathbb{N}} \in r(X) : \\ \inf_{n\in\mathbb{N}} \sup_{m\geq n} \mu(x_m) \leq \lambda\{x_n\}_{n\in\mathbb{N}}(x).$$

A good definition of (approach) closure-sequential spaces will give us, for topological approach spaces, the topological definition of closure-sequential spaces.

PROPOSITION 3.2. *A topological space $(X, \mathcal{T})$ is closure-sequential in* **TOP** *if and only if X is closure-sequential in* **AP**.

Proof. Suppose $(X, \mathcal{T})$ is sequential and consider $\mu \in [0,\infty]^X$ for which,

$$\inf_{n\in\mathbb{N}} \sup_{m\geq n} \mu(x_m) \leq \lambda\{x_n\}_{n\in\mathbb{N}}(x) \tag{3.1}$$

for all $\{x_n\}_{n\in\mathbb{N}} \in r(X)$.
Consider, $\forall\varepsilon > 0$, the set $U_\varepsilon := \{x \in X \mid \mu(x) < \varepsilon\}$ and suppose a sequence $\{x_n\}_{n\in\mathbb{N}}$ has a point x in U_ε as a limit point. It follows from (3.1) that

$$\inf_{n\in\mathbb{N}} \sup_{m\geq n} \mu(x_m) = 0$$

which means that $\{x_n\}_{n\in\mathbb{N}}$ is eventually in U_ε.
Consequently, U_ε is sequential neighbourhood of the point x and, further, a neighbourhood of x. Now, since $\mu - \varepsilon \leq 1_{U_\varepsilon}$, we obtain $\mu - \varepsilon \in \phi(x)$. Because of the arbitrariness of ε and of the saturation of $\phi(x)$, it follows that $(X, \mathcal{A}_\mathcal{T})$ is closure-sequential.
Conversely, suppose X is closure-sequential (as an approach space) and

consider $U \subset X$ sequential neighbourhood of a point $x \in X$. To see that U is also a neighbourhood of x, we put

$$\mu = \theta_U \in [0, \infty]^X.$$

Then if, for an arbitrary sequence $\{x_n\}_{n \in \mathbb{N}}$, we have $\lambda\{x_n\}_{n \in \mathbb{N}}(x) = 0$, it will follow that x is a limit point of $\{x_n\}_{n \in \mathbb{N}}$ and, thus, $\{x_n\}_{n \in \mathbb{N}}$ is eventually in U which turns into

$$\inf_{n \in \mathbb{N}} \sup_{m \geq n} \mu(x_m) = 0$$

Upon invoking 2.2.8 in [11], the fact that X is a closure-sequential approach space means that U is a neighbourhood of x (in $\mathcal{T}$). □

It is important to know, from an structural point of view, the following results:

PROPOSITION 3.3. *Every first countable approach space is a closure-sequential space.*

Proof. Suppose X has a countable basis for each of the ideals $\mathcal{A}(x)$ of local distances,

$$\mathcal{A}(x) = < \{\varphi_1,\ \varphi_2,\ ...,\ \varphi_l ...\}_{l \in \mathbb{N}} > .$$

We replace φ_l by $\vee_{k=1}^{l} \varphi_k$ whenever necessary and we may assume that $\{\varphi_l\}_{l \in \mathbb{N}}$ is increasing.
Consider $x \in X$, $A \subset X$. Then

$$\delta(x, A) = \sup_{l \in \mathbb{N}} \inf_{y \in A} \varphi_l(y)$$

which means that, for all $n \in \mathbb{N}$, we can find $l_n \in \mathbb{N}$ with

$$\inf_{y \in A} \varphi_{l_n}(y) > \delta(x, A) - 1/n.$$

This allows us to choose $x_n \in A$ with

$$\delta(x, A) \geq \varphi_{l_n}(x_n) > \delta(x, A) - 1/n.$$

Now clearly,

$$\begin{aligned}
\lambda\{x_n\}_{n \in \mathbb{N}}(x) &= \sup_{l \in \mathbb{N}} \inf_{n \in \mathbb{N}} \sup_{m \geq n} \varphi_l(x_m) \\
&\leq \sup_{l \in \mathbb{N}} \inf_{n \in \mathbb{N}} \sup_{m \geq n} \varphi_{l_m}(x_m) \\
&= \inf_{n \in \mathbb{N}} \sup_{m \geq n} \varphi_{l_m}(x_m) \\
&= \delta(x, A).
\end{aligned}$$

With this we can conclude that, in a first-countable approach space (X, δ), given $x \in X$ and $A \subset X$, it is always possible to find a sequence $\{x_n\}_{n\in\mathbb{N}}$ within A with the property

$$\delta(x, A) = \lambda\{x_n\}_{n\in\mathbb{N}}(x).$$

Next we take $\mu \in [0, \infty]^X$.
If, for all $\{z_n\}_{n\in\mathbb{N}} \in r(X)$ the following holds

$$\inf_{n\in\mathbb{N}} \sup_{m\geq n} \mu(z_m) \leq \lambda\{z_n\}_{n\in\mathbb{N}}(x),$$

then, in particular,

$$\inf_{n\in\mathbb{N}} \sup_{m\geq n} \mu(x_m) \leq \delta(x, A),$$

from which it follows

$$\inf_{y\in A} \mu(y) \leq \delta(x, A),$$

in other words, that $\mu \in \mathcal{A}(x)$. $\square$

Theorem 3.4. **CSEQ** *(the category of closure-sequential approach spaces) is embedded as a concretely coreflective subconstruct of* **AP**. *For any approach space* $X, \mathcal{A})$*, its* **CSEQ***-bicoreflection is determined by the following collection of ideals as approach system: given* $x \in X$*,*

$$\Phi^{ccs}(x) := (\{\varphi \in [0, \infty]^X \mid \forall \{x_n\}_{n\in\mathbb{N}} \in r(X) \ : \\ \inf_{n\in\mathbb{N}} \sup_{m\geq n} \varphi(x_m) \leq \lambda\{x_n\}_{n\in\mathbb{N}}(x)\})$$

where λ *is the limit operator for the approach space* $(X, \mathcal{A})$.

Proof. It is easily verified that $(\Phi^{ccs}(x))_{x\in X}$ is indeed an approach system for a closure-sequential (approach) space and that $id_X \ : \ (X, \Phi^{ccs}) \to (X, \mathcal{A})$ is a contraction. Now, suppose that $(Y, \Phi) \in |\mathbf{CSEQ}|$ and that

$$f \ : \ (Y, \Phi) \ \to \ (X, \mathcal{A})$$

is a contraction. We consider $\varphi \in \Phi^{ccs}(f(y))$. Then, given $\{y_n\}_{n\in\mathbb{N}} \in r(Y)$,

$$\begin{aligned} \inf_{n\in\mathbb{N}} \sup_{m\geq n} \varphi(\{f(y_m)\}_{m\in\mathbb{N}}) &\leq \lambda(\{f(y_n\}_{n\in\mathbb{N}})(f(y)) \\ &\leq \lambda^Y(\{y_n\}_{n\in\mathbb{N}})(y) \end{aligned}$$

which, since Y is a closure-sequential space, means that $\varphi \circ f \in \Phi(y)$. $\square$

As is well known, countable compactness and sequential compactness coincide for first-countable approach spaces. Since sequential compactness is always bigger or equal than countable compactness, the following proposition establishes their equality on the larger category of closure -sequential approach spaces

Proposition 3.5. *For a closure-sequential approach space* $(X, (\mathcal{A}(x))_{x\in X})$ *the following holds:*

$$\mu_{sc}(X) = \mu_{cc}(X).$$

Proof. Suppose $\varepsilon_0 > 0$ can be found such that

$$\mu_{sc}(X) > \mu_{cc}(X) + \varepsilon_0.$$

This means there exists a sequence $\{x_n^0\}_{n\in\mathbb{N}}$ for which

$$\inf_{x\in X} \lambda\{x_{k(n)}^0\}_{n\in\mathbb{N}}(x) \ > \ \inf_{x\in X} \alpha\{x_n^0\}_{n\in\mathbb{N}}(x) + \varepsilon_0$$

and this holds for every $\{x_{k(n)}^0\}_{n\in\mathbb{N}}$, subsequence of $\{x_n^0\}_{n\in\mathbb{N}}$.
We can then find a point $x_0 \in X$ such that, for any other point x_n^0 in the sequence $\{x_n^0\}_{n\in\mathbb{N}}$,

$$\sup_{\varphi\in\mathcal{A}(x_0)} \varphi(x_n^0) > \sup_{\varphi\in\mathcal{A}(x_0)} \sup_{n\in\mathbb{N}} \inf_{m\geq n} \varphi(x_n^0) + \varepsilon_0$$

which allows us to choose $\varphi_n \in \mathcal{A}(x_0)$, for each $n \in \mathbb{N}$, with

$$\varphi_n(x_n^0) > \sup_{l\in\mathbb{N}} \inf_{m\geq l} \varphi_n(x_m^0) + \varepsilon_0.$$

Next we define $\mu \in [0, \infty]^X$ in the following way

$$\mu(y) = \begin{cases} \varphi_n(x_n^0), & \text{if } y = x_n^0 \text{ element of } \{x_n^0\}_{n\in\mathbb{N}}; \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for a given $\{y_n\}_{n\in\mathbb{N}} \in r(X)$, we first observe that $\inf_{n\in\mathbb{N}} \sup_{m\geq n} \mu(y_m) > 0$ if, and only if, $\forall n \in \mathbb{N}$, $\exists m_n \geq n$ such that y_{m_n} is an element of the sequence $\{x_n^0\}_{n\in\mathbb{N}}$.
We denote $\{z_l^n\}_{l\in\mathbb{N}}$ the subsequence of $\{x_n^0\}_{n\in\mathbb{N}}$ for which, for each $l \in \mathbb{N}$, there exists $l_n \geq n$ with $y_{l_n} = z_l^n$.

Then, as a consequence of the definition of μ

$$\begin{aligned}\inf_{n\in\mathbb{N}}\sup_{m\geq n}\mu(y_m) &= \inf_{n\in\mathbb{N}}\sup_{l\in\mathbb{N}}\mu(y_{l_n})\\ &\leq \sup_{\theta\in\mathcal{A}(x_0)}\inf_{n\in\mathbb{N}}\sup_{l\in\mathbb{N}}\theta(y_{l_n})\\ &\leq \sup_{\theta\in\mathcal{A}(x_0)}\inf_{n\in\mathbb{N}}\sup_{m\geq n}\theta(y_m)\\ &= \lambda\{y_n\}_{n\in\mathbb{N}}(x_0).\end{aligned}$$

We are now in the situation to conclude that $\mu \in \mathcal{A}(x_0)$, since X is a closure-sequential approach space. But that would easily implies:

$$\begin{aligned}\alpha\{x_n^0\}_{n\in\mathbb{N}}(x_0) &\geq \sup_{l\in\mathbb{N}}\inf_{m\geq l}\mu(x_m^0)\\ &= \sup_{l\in\mathbb{N}}\inf_{m\geq l}\varphi_m(x_m^0)\\ &> \sup_{l\in\mathbb{N}}\inf_{m\geq l}\alpha\{x_n^0\}_{n\in\mathbb{N}}(x_0)+\varepsilon_0\\ &= \alpha\{x_n^0\}_{n\in\mathbb{N}}(x_0)+\varepsilon_0.\end{aligned}$$

□

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